2

FATIGUE DAMAGE THEORIES

YUNG-LI LEE
DAIMLERCHRYSLER

2.1 INTRODUCTION

Predicting fatigue damage for structural components subjected to variable loading conditions is a complex issue. The first, simplest, and most widely used damage model is the linear damage. This rule is often referred to as Miner's rule (1945). However, in many cases the linear rule often leads to nonconservative life predictions. The results from this approach do not take into account the effect of load sequence on the accumulation of damage due to cyclic fatigue loading. Since the introduction of the linear damage rule many different fatigue damage theories have been proposed to improve the accuracy of fatigue life prediction. A comprehensive review of many fatigue damage approaches can be found elsewhere (Fatemi and Yang, 1998). This chapter addresses (1) underlying fatigue damage mechanisms, (2) fatigue damage models commonly used in the automotive industry, and (3) postulations and practical implementations of these damage rules.

2.2 FATIGUE DAMAGE MECHANISM

Fatigue is a localized damage process of a component produced by cyclic loading. It is the result of the cumulative process consisting of crack initiation, propagation, and final fracture of a component. During cyclic loading, localized plastic deformation may occur at the highest stress site. This plastic deformation induces permanent damage to the component and a crack develops. As the component experiences an increasing number of loading cycles, the length of the crack (damage) increases. After a certain number of cycles, the crack will cause the component to fail (separate).

In general, it has been observed that the fatigue process involves the following stages: (1) crack nucleation, (2) short crack growth, (3) long crack growth, and (4) final fracture. Cracks start on the localized shear plane at or near high stress concentrations, such as persistent slip bands, inclusions, porosity, or discontinuities. The localized shear plane usually occurs at the surface or within grain boundaries. This step, crack nucleation, is the first step in the fatigue process. Once nucleation occurs and cyclic loading continues, the crack tends to grow along the plane of maximum shear stress and through the grain boundary.

A graphical representation of the fatigue damage process shows where crack nucleation starts at the highest stress concentration site(s) in the persistent slip bands (Figure 2.1). The next step in the fatigue process is the crack growth stage. This stage is divided between the growth of Stage I and Stage II cracks. Stage I crack nucleation and growth are usually considered to be the initial short crack propagation across a finite length of the order of a couple of grains on the local maximum shear stress plane. In this stage, the crack tip plasticity is greatly affected by the slip characteristics, grain size, orientation, and stress level, because the crack size is comparable to the material microstructure. Stage II crack growth refers to long crack propagation normal to the principal tensile stress plane globally and in the maximum shear stress direction locally. In this stage, the characteristics of the long crack are less affected by the properties of the microstructure than the Stage I crack. This is because the crack tip plastic zone for Stage II crack is much larger than the material microstructure.

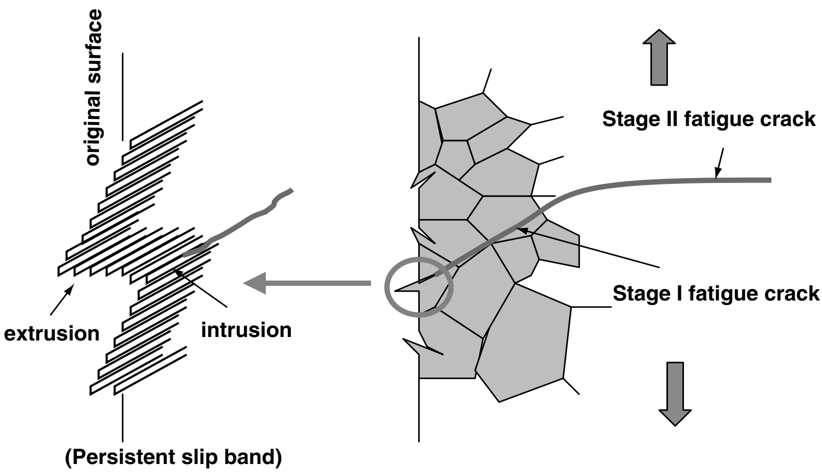

FIGURE 2.1 The fatigue process: a thin plate under cyclic tensile loading.

In engineering applications, the amount of component life spent on crack nucleation and short crack growth is usually called the *crack initiation period*, whereas the component life spent during long crack growth is called the *crack propagation period*. An exact definition of the transition period from initiation to propagation is usually not possible. However, for steels the size of a crack at the end of the initiation stage, a_0, is of the order of a couple of grains of the material. This crack size typically ranges from about 0.1 to 1.0 mm. The crack initiation size can be estimated using the linear elastic fracture mechanics approach for smooth specimens by Dowling (1998):

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th}}{\Delta S_e}\right)^2 \tag{2.2.1}$$

or by 0.1 to 0.2 times the notch-tip radius for notched specimens (Dowling, 1998), or by twice the Peterson empirical material constant for steels (Peterson, 1959):

$$a_0(\text{mm}) = 2 \times 0.0254 \times \left(\frac{2079}{S_u(\text{MPa})}\right)^{1.8} \tag{2.2.2}$$

where S_u is the ultimate tensile strength of a material, ΔS_e is the stress range at the fatigue limit, and ΔK_{th} is the range of the threshold intensity factor for $R = -1$.

Typically, the crack initiation period accounts for most of the fatigue life of a component made of steels, particularly in the high-cycle fatigue regime (approximately >10,000 cycles). In the low-cycle fatigue regime (approximately <10,000 cycles), most of the fatigue life is spent on crack propagation.

Once a crack has formed or complete failure has occurred, the surface of a fatigue failure can be inspected. A bending or axial fatigue failure generally leaves behind clamshell or beach markings. The name for these markings comes from the appearance of the surface. An illustration of these markings is shown in Figure 2.2. The crack nucleation site is the center of the shell, and the crack appears to propagate away from the nucleation site, usually in a radial manner. A semielliptical pattern is left behind. In some cases, inspection of the size and location of the beach marks left behind may indicate where a different period of crack growth began or ended.

Within the beach lines are striations. The striations shown in Figure 2.2 appear similar to the rings on the cross-section of a tree. These striations represent the extension of the crack during one loading cycle. Instead of a ring for each year of growth, there is a ring for each loading cycle. In the event of a failure, there is a final shear lip, which is the last bit of material supporting the load before failure. The size of this lip depends on the type of loading, material, and other conditions.

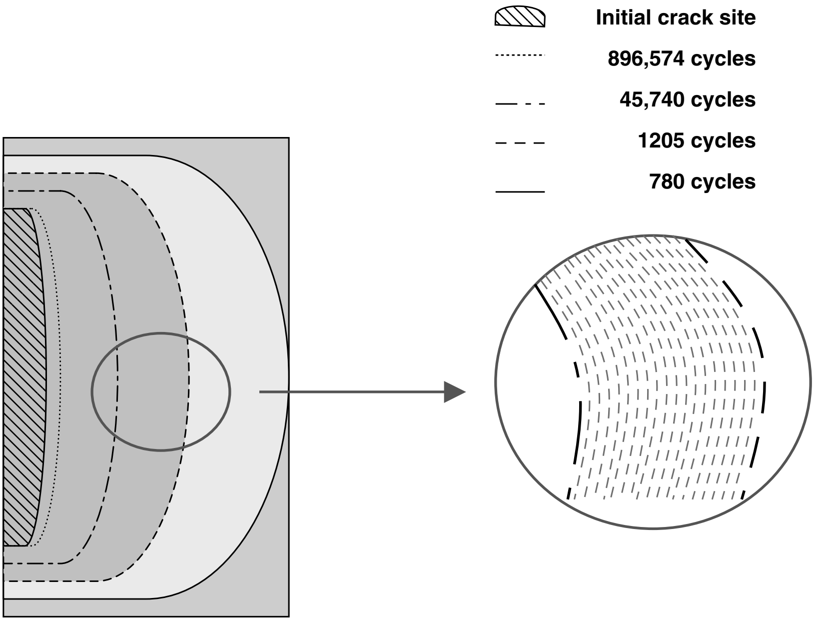

FIGURE 2.2 Fracture surface markings and striations.

2.3 CUMULATIVE DAMAGE MODELS—THE DAMAGE CURVE APPROACH

The component's damage can be expressed in terms of an accumulation of the crack length toward a maximum acceptable crack length. For example, a smooth specimen with a crack length at fracture of a_f is subjected to cyclic loading that results in a crack length of a. The amount of damage, D, at a given stress level S_1, would be the ratio of a to a_f. To illustrate the cumulative damage concept, a crack growth equation developed by Manson and Halford (1981) is adopted:

$$a = a_0 + (a_f - a_0)\left(\frac{n}{N_f}\right)^{\alpha_f} \tag{2.3.1}$$

Equation 2.3.1 was derived based on early crack growth fracture mechanics and fitted with a large amount of test data for loading with two-step life/stress levels, where n is the number of loading cycles applied to achieve a crack length of a, and a_0 is the initial crack length. The value N_f represents the number of cycles applied to achieve the crack length a_f at final fracture. The exponent α_f is empirically determined and has the following form:

$$\alpha_f = \frac{2}{3} N_f^{0.4} \tag{2.3.2}$$

Cumulative damage (D) is the ratio of instantaneous to final crack length and can be expressed as follows:

$$D = \frac{a}{a_f} = \frac{1}{a_f}\left[a_0 + (a_f - a_0)\left(\frac{n}{N_f}\right)^{\alpha_f}\right] \tag{2.3.3}$$

This damage equation implies that fatigue failure occurs when D is equal to unity (i.e., $a = a_f$).

Consider a two-step high–low sequence loading in Figure 2.3, where n_1 denotes the initial applied load cycles with a higher stress or load level and $n_{2,f}$ the remaining cycles to eventual fatigue failure with a lower stress or load level. Note that the subscripts 1 and 2 refer to sequence of the applied loading: 1 is the first and 2 is the second load level. The S–N curve is used to obtain the fatigue lives $N_{1,f}$ and $N_{2,f}$ for each load level. Nonlinear damage curves for two different loads are shown schematically in Figure 2.4. Each of these curves represents a different loading condition that leads to a different time to failure (or life level). At each load level or life level, the relation between the damage value and the applied cycles or the cycle ratio follows the power law equation in Equation 2.3.3. If a cycle ratio $n_1/N_{1,f}$ is first applied along the curves representing the life level $N_{1,f}$ to point OA, the damage accumulation process will be represented by the life level curve $N_{1,f}$ from zero to point A. If at this point a new loading level with a life of $N_{2,f}$ is introduced and this loading is applied, the damage process will proceed from point A to point A′ from the same damage value.

If the load corresponding to the cycle ratio $n_{2,f}/N_{2,f}$ is applied from A′ to B′ at the life level N_{2f}, failure takes place if $D = 1.0$ is reached at point B′. From this figure, it is clear that if a higher load level with a lower life along OA is first applied and followed by the lower load magnitude with a higher

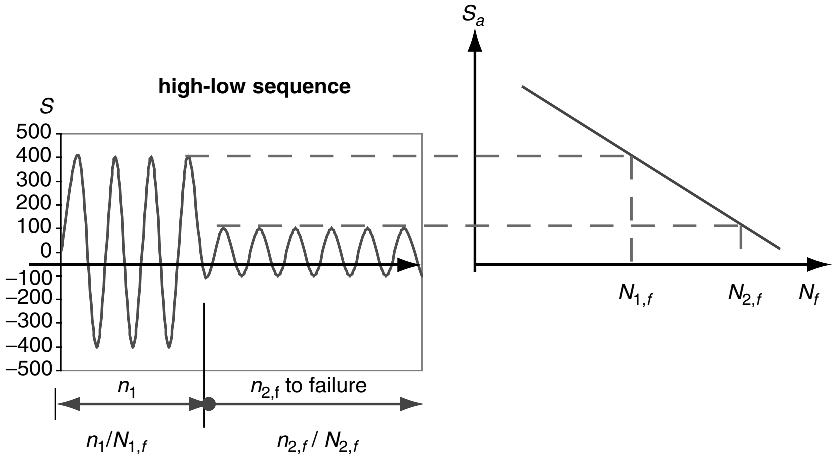

FIGURE 2.3 A block of two-step high–low sequence loading.

life along A′B′, the sum of cycle ratios will be smaller than unity. Thus, the estimated fatigue life depends on the sequence of loading. However, if a lower load level is applied first along OA′ and is followed by the higher load level along AB, the summation of the cycle ratios is greater than unity because the cycle ratio AA′ is accounted for twice.

Based on the equal damage at A and A′ in Figure 2.4 for the two load levels, the following equations hold true for the relation between the cycle ratio $n_1/N_{1,f}$ and the equivalent damage cycle ratio $n_2/N_{2,f}$:

$$\left(\frac{n_{2,f}}{N_{2,f}}\right) = 1 - \left(\frac{n_1}{N_{1,f}}\right)^{\left(N_{1,f}/N_{2,f}\right)^{0.4}} \tag{2.3.4}$$

and

$$\frac{n_1}{N_{1,f}} = \left[\frac{n_2}{N_{2,f}}\right]^{\left(N_{2,f}/N_{1,f}\right)^{0.4}} \tag{2.3.5}$$

where n_2 is the number of cycles at the life level $N_{2,f}$, equivalent damage to the initial cycle ratio $n_1/N_{1,f}$.

It is clear that Equation 2.3.5 is independent of material and geometric parameters (e.g., a_0, a_f, and α_f) that were introduced in the damage accumulation equation (Equation 2.3.3). Thereby, a nonlinear damage curve for a reference life level ($N_{1,f}$) can be linearized by replacing a_0 by zero and 2/3 by $\left(1/N_{1,f}\right)^{0.4}$. Thus, the damage function for the reference life level can be simplified as a linear line connecting (0,0) with (1,1), i.e.,

$$D_1 = \frac{n_1}{N_{1,f}} \tag{2.3.6}$$

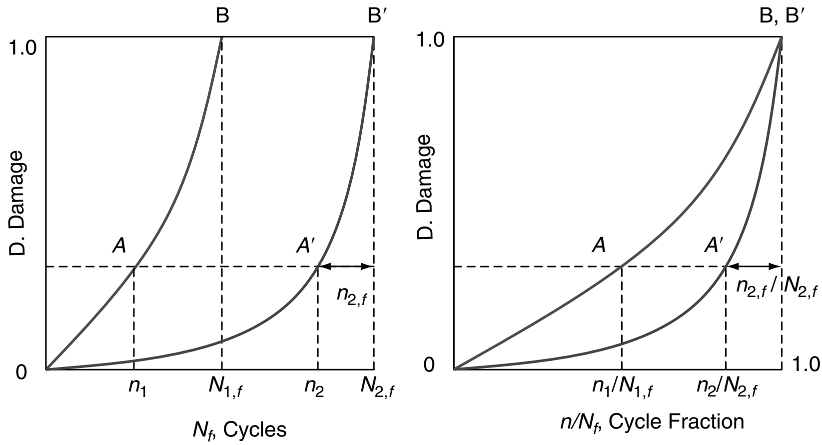

FIGURE 2.4 Nonlinear damage accumulation.

Therefore, the damage curve for another life level ($N_{2,f}$) is then given by the power law damage equation defined as:

$$D_2 = \left[\frac{n_2}{N_{2,f}}\right]^{\left(N_{2,f}/N_{1,f}\right)^{0.4}} \tag{2.3.7}$$

For multiple life levels (e.g., $N_{1,f} < N_{2,f} < \cdots < N_{n,f}$), the damage curves can be constructed expeditiously by letting the damage curve for the lowest life level be the reference life.

Two other methods have been developed to determine the power law damage equation. Based on the experimental observations that the equivalent damage S–N lines converge to the fatigue limit, Subramanyan (1976) calculated damage by referring to the reference stress amplitude S_{ref} and the fatigue limit S_e as follows:

$$D_N = \left[\frac{n_n}{N_{n,f}}\right]^{(S_n - S_e)/(S_{ref} - S_e)} \tag{2.3.8}$$

On the other hand, Hashin (1980) expressed this differently by using the fatigue life N_e at the fatigue limit S_e:

$$D_N = \left[\frac{n_n}{N_{n,f}}\right]^{\log(N_{n,f}/N_e)/\log(N_{ref}/N_e)} \tag{2.3.9}$$

The power law accumulation theories using the three different methods are compared with the experimental data by Manson et al. (1967). The high-to-low (H-L) step-stress series was applied to smooth components made of SAE 4130 steel with a soft heat treatment. The two stress levels are 881 and 594 MPa, which correspond to fatigue lives of 1700 and 81,250 cycles, respectively. A fatigue limit of 469 MPa was also determined at 800,000 cycles. By applying Equations 2.3.7–2.3.9, the three power law exponent values are obtained (i.e., 0.213 on Manson and Halford, 0.303 on Subramanyan, and 0.372 on Hashin). Figure 2.5 compares the predicted fatigue behavior of the three power law damage rules with experimental data for SAE 4130. The curve derived from the Mason rule is close to the experimental data and the Subramanyan and the Hashin curves are slightly nonconservative.

Example 2.1. An unnotched component is subjected to a four-level step-stress fatigue test, which starts with the highest load level, ± 800 MPa, to the lowest level, ± 200 MPa. At each load level, a cycle fraction of 0.01 is added before proceeding to the next level. The sequence 1, 2, 3, 4 is repeated until failure occurs when $D = 1.0$. The specific four-level step stresses (S_i), the applied cycles (n_i), and the associated fatigue lives ($N_{i,f}$) are listed in Table 2.1. Estimate the fatigue life of the specimen based on the damage curve approach by Manson and Halford.

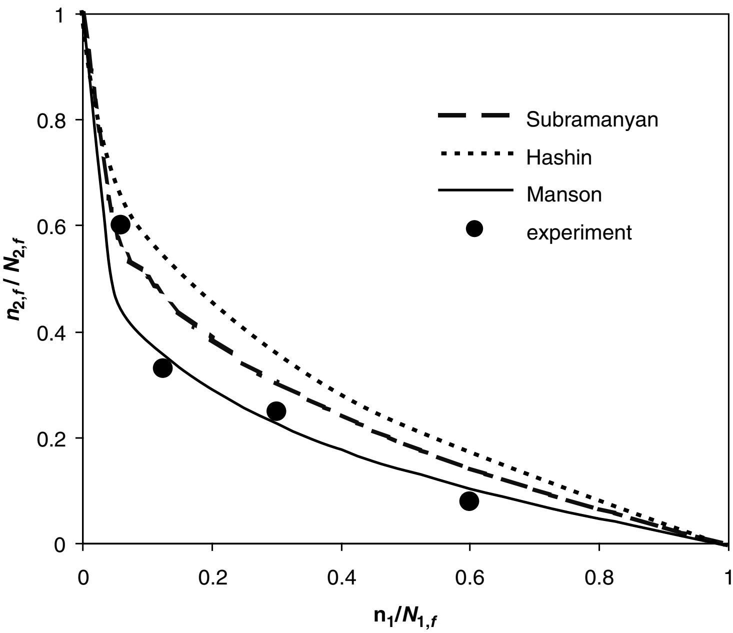

FIGURE 2.5 Comparison of predicted fatigue behavior with experimental data for SAE 4130 steel for two-step loading.

TABLE 2.1 Four-Level Step-Stress Fatigue Test Data

Four-Level Step-Stress S_i (Mpa)	n_i (cycles)	$N_{i,f}$ (cycles)
±800	10	1000
±600	100	10,000
±400	1000	100,000
±200	10,000	1,000,000

Solution. Choose $N_{ref} = 10^3$ cycles for the reference life because it has the shortest life of the four life levels and so the linear damage rule applies for this life level. For the first block of loading, proceed along the straight line curve until $D_1 = 10/10^3 = 0.01$. Then stop and traverse horizontally until on the damage curve for $N_{2,f} = 10^4$ cycles at $D_1 = 0.01$. The equivalent damage cycles for $N_{2,f} = 10^4$ cycles are $n_{2,eq} = N_{2,f} \times [D_1]^{(N_{ref}/N_{2,f})^{0.4}} = 10^4 \times (0.01)^{(10^3/10^4)^{0.4}} = 1599$ cycles.

With the additional $n_2 = 10^2$ cycles applied in the second block of loading, the accumulated damage value becomes

$$D_{1+2} = \left[\frac{n_{2,eq} + n_2}{N_{2,f}}\right]^{\left[N_{2,f}/N_{ref}\right]^{0.4}} = \left[\frac{1599 + 100}{10^4}\right]^{\left[10^4/10^3\right]^{0.4}} = 0.01165$$

Proceed horizontally to the next damage curve for $N_{3,f} = 10^5$ cycles at $D_{1+2} = 0.01165$. The corresponding number of cycles are

$$n_{3,eq} = N_{3,f} \times [D_{1+2}]^{(N_{ref}/N_{3,f})^{0.4}} = 10^5 \times (0.01165)^{(10^3/10^5)^{0.4}} = 49378 \text{ cycles}$$

By adding $n_3 = 10^3$ cycles in the third block of loading, the damage accumulation so far is

$$D_{1+2+3} = \left[\frac{n_{3,eq} + n_3}{N_{3,f}}\right]^{[N_{3,f}/N_{ref}]^{0.4}} = \left[\frac{49378 + 1000}{10^5}\right]^{[10^5/10^3]^{0.4}} = 0.01322$$

Finally, move horizontally to the last damage curve for $N_{4,f} = 10^6$ cycles at $D_{1+2+3} = 0.01322$. The equivalent damage cycles are

$$n_{4,eq} = N_{4,f} \times [D_{1+2+3}]^{(N_{ref}/N_{4,f})^{0.4}} = 10^6 \times (0.01322)^{(10^3/10^6)^{0.4}} = 761{,}128$$

The accumulated damage for additional $n_4 = 10^4$ cycles is calculated:

$$D_{1+2+3+4} = \left[\frac{n_{4,eq} + n_4}{N_{4,f}}\right]^{[N_{4,f}/N_{ref}]^{0.4}} = \left[\frac{761128 + 10000}{10^6}\right]^{[10^6/10^3]^{0.4}} = 0.01626$$

This completes the calculation of the total damage accumulation for the four-level step loading. Then traverse horizontally back to the first curve where you move up along the linear line from the damage value of 0.01626 to 0.02626 before going back horizontally to the second curve, etc. Advance alternately up these curves until a total damage value of $D = 1$ is eventually reached. The complete list of iterations is illustrated in Figure 2.6 and Table 2.2. In this example, it takes 11 blocks to reach the failure point.

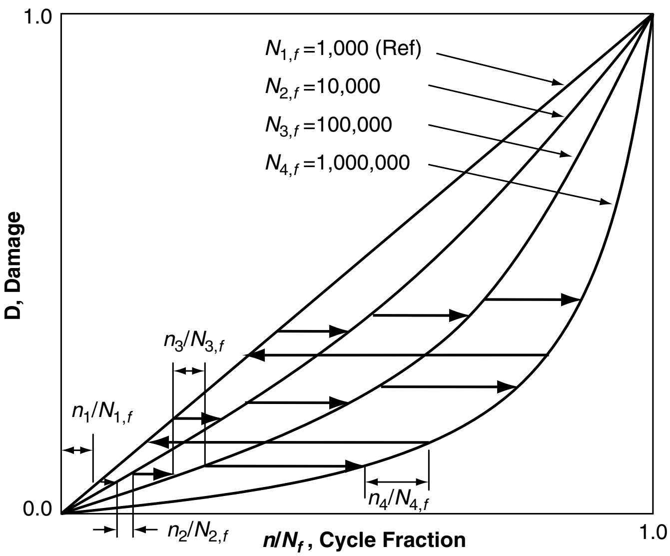

FIGURE 2.6 An example of DCA for four-step loading.

TABLE 2.2 Procedures for Fatigue Life Estimations Based on the Manson and Halford Damage Curve Approach

No. of Blocks	$n_{eq} + n_1$	D_1	$n_{eq} + n_2$	D_{1+2}	$n_{eq} + n_3$	D_{1+2+3}	$n_{eq} + n_4$	$D_{1+2+3+4}$
1	10	0.01000	1699	0.01165	50378	0.01322	771128	0.01625
2	26	0.02625	2448	0.02915	58105	0.03253	815621	0.03955
3	50	0.04955	3123	0.05377	63923	0.05940	846818	0.07170
4	82	0.08170	3789	0.08738	68956	0.09582	872450	0.11503
5	125	0.12503	4470	0.13234	73577	0.14427	895010	0.17239
6	182	0.18239	5179	0.19155	77957	0.20780	915622	0.24731
7	257	0.25731	5925	0.26855	82191	0.29012	934891	0.34403
8	354	0.35403	6714	0.36763	86334	0.39567	953177	0.46765
9	478	0.47765	7552	0.49392	90423	0.52982	970713	0.62432
10	634	0.63432	8443	0.65354	94482	0.69898	987656	0.82131
11	831	0.83131	9391	0.85397	98529	0.91074	1004118	1.06730

2.4 LINEAR DAMAGE MODELS

If the damage curves in the cycle $N_{i,f}$ coordinate are linearized, the linear damage rule is developed by reducing the damage curves to a single line in the cycle ratio $n_i/N_{i,f}$ domain, as shown in Figure 2.7. In this case, the fatigue damage has a unique, linear relation with the cycle ratio regardless of the stress levels. Hence, at a given level of damage, the cycle ratio for two different damage curves will be the same. This is illustrated by Figure 2.7, in which two linear damage curves plotted on a graph of damage versus cycles to failure are equal to each other when plotted on a graph of damage versus cycle ratio. Failure will occur when the sum of the ratios at each stress

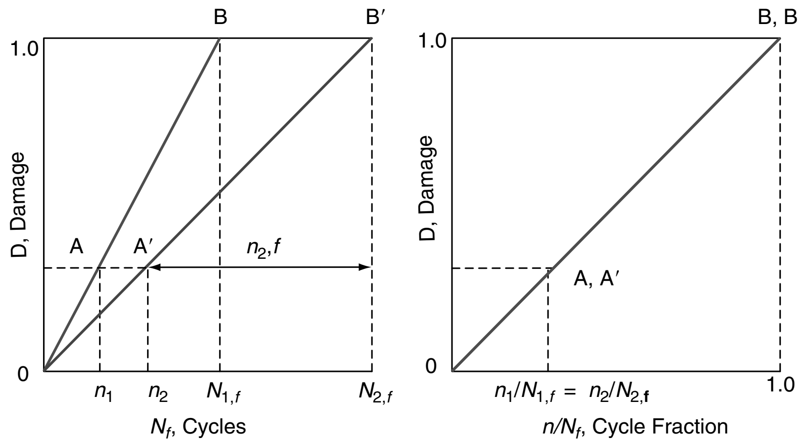

FIGURE 2.7 Linear damage accumulation.

level reaches a value of 1.0. In terms of mathematics, the linear damage rule can be expressed as follows:

$$D_i = \frac{n_i}{N_{i,f}} \tag{2.4.1}$$

Failure is predicted when

$$\Sigma D_i = \sum \frac{n_i}{N_{i,f}} \geq 1.0 \tag{2.4.2}$$

The universally used linear damage assessment model was first proposed by Palmgren (1924) for application to the Swedish ball bearing industry. Langer (1937), working for Westinghouse in the electric power generation area, independently proposed a similar linear rule for pressure vessel and piping components made of steel. Miner (1945) of Douglas Aircraft built on Langer's work and applied the linear damage rule to tension–tension axial fatigue data for aircraft skin material (aluminum alloy 24S-T ALCLAD). Miner demonstrated excellent agreement between the predictions from the linear damage rule and his experimental results. This success led to the strong association between Miner and the linear damage rule, and the linear damage rule is commonly referred to as Miner's linear damage rule.

Since Miner's work was conducted, the linear damage rule has been demonstrated to be unreliable. Studies by Wirshing et al. (1995) and Lee et al. (1999) are listed in Table 2.3 and show that the median damage values to test specimens under certain loading conditions range from 0.15 to 1.06. This is attributable to the fact that the relationship between physical damage (i.e., crack size or crack density) and cycle ratio is not unique and varies from one stress level to another.

TABLE 2.3 Statistics on Damage Values

	Median	Coefficient of Variation	Statistical Distribution
Miner: Original work	0.95	0.26	Lognormal
Schutz: Crack initiation			
29 Random sequence test series	1.05	0.55	Lognormal
Test with large mean load change	0.60	0.60	Lognormal
Significant notch plastic strain	0.37	0.78	Lognormal
Automotive axle spindle	0.15	0.60	Lognormal
Shin and Lukens: Extensive random test data	0.90	0.67	Lognormal
Gurney: Test data on welded joints	0.85	0.28	Lognormal
Lee: Mean stress effect—SAE cumulative fatigue test data	1.06	0.47	Normal

Example 2.2. Repeat Example 2.1 to determine the damage value for the four-level step loading, using the linear damage rule.

Solution. The damage value due to the four-level step stresses is

$$\sum D_i = \frac{n_1}{N_{1,f}} + \frac{n_2}{N_{2,f}} + \frac{n_3}{N_{3,f}} + \frac{n_4}{N_{4,f}} = \frac{10}{10^3} + \frac{10^2}{10^4} + \frac{10^3}{10^5} + \frac{10^4}{10^6} = 0.04$$

The total number of repetitions required for the four-step loading to reach the fatigue failure point can be determined as follows:

$$\text{Repetitions} = \frac{1.0}{\sum D_i} = \frac{1.0}{0.04} = 25$$

The estimated 25 repetitions based on the linear damage rule are nonconservative compared with the estimated 11 repetitions from the nonlinear damage theory.

2.5 DOUBLE LINEAR DAMAGE RULE BY MANSON AND HALFORD

The tedious iteration process using the nonlinear damage theory and the deficiency in damage assessment using the simple linear damage rule have motivated researchers to look for a better way to overcome the disadvantages of each method. Based on the observation that fatigue is at least a two-phase process—crack initiation and crack propagation—the models for the damage curves can be assumed to be bilinear.

Examples of such curves are shown in Figure 2.8. The bilinear model represents an equivalent damage model to the damage curve accumulation rule.

Manson and Halford (1981) derived the required criteria to determine the coordinates of the knee point, i.e., the intersection between the two straight lines of the bilinear curves. It is suggested that the straight line connecting (0, 0) and (1, 1) be the reference damage line for the lowest life level. Because of the nonlinear nature of damage and the accumulation of damage being modeled as a bilinear process, the two regions of damage are identified. The region of damage from the origin to the level of AA′ is designated as Phase I (D_I), and the region of damage from AA′ to BB′ is defined as Phase II (D_{II}). Using an approach similar to the method presented to normalize the cycle ratio for linear damage accumulation, the cycle ratios for Phase I and Phase II damage accumulations are constructed in a linear fashion, as shown in Figure 2.9.

Figure 2.9 illustrates that the total damage can be decomposed into Phase I damage (D_I) and Phase II damage (D_{II}). The Phase I linear damage

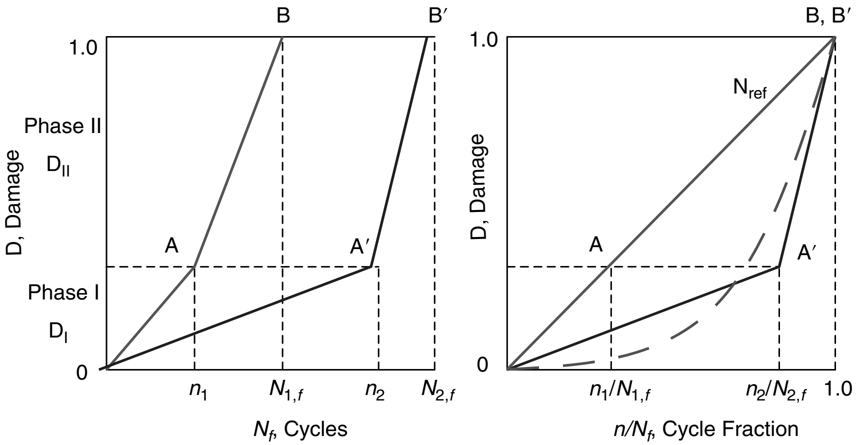

FIGURE 2.8 Double linear damage accumulation.

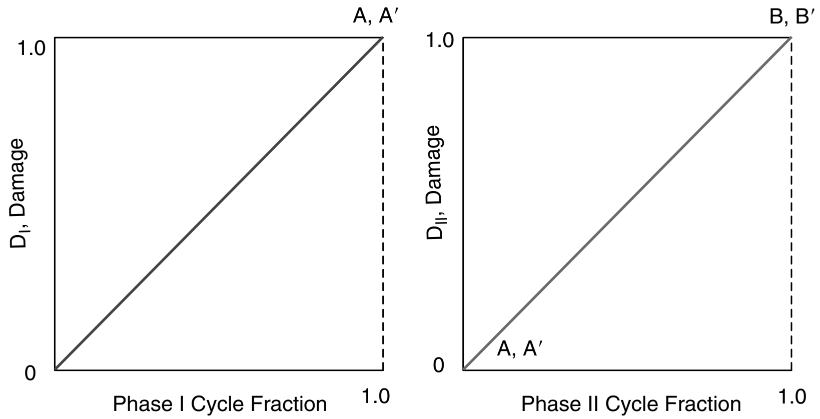

FIGURE 2.9 Phase I and Phase II linear damage rules.

accumulation rule states that prior to reaching the damage D_I, the cycle ratios can be summed linearly and are independent of the loading sequence (i.e., from OA to OA′ or from OA′ to OA). When the sum of the cycle ratios reaches unity, Phase I damage is completed. After the total damage beyond D_I, the Phase II linear damage accumulation rule applies. Regardless of the loading sequence, the damage accumulation depends only on the total sum of cycle fractions at each level. Based on the considerable amount of test data developed for two-step loading on many materials, Manson and Halford discovered that the knee point between Phase I and Phase II damage depends on the ratio of $N_{1,f}/N_{2,f}$ instead of the physical significance of crack initiation and crack propagation. Figure 2.10 shows the linear damage rule for the H-L step stress loading with the initial applied cycle fraction $n_1/N_{1,f}$ and the remaining cycle fraction $n_{2,f}/N_{2,f}$. Figure 2.10 illustrates the double linear damage rule and damage curve accumulations for the two-step loading and the relationship between $n_1/N_{1,f}$ and $n_{2,f}/N_{2,f}$. Equation 2.3.4 is the mathematical model for the description of the relationship between $n_1/N_{1,f}$ and $n_{2,f}/N_{2,f}$. To meet the condition that the bilinear model is equivalent to the damage curve accumulation rule, the knee point coordinates were derived and are found to depend on the ratio of $N_{1,f}/N_{2,f}$. Figure 2.11 shows that the test data on the knee point coordinates correlates well with $N_{1,f}/N_{2,f}$. The coordinates of the knee point are empirically determined as follows:

$$\left[\frac{n_1}{N_{1,f}}\right]_{\text{knee}} = 0.35 \times \left(\frac{N_{1,f}}{N_{2,f}}\right)^{0.25} \tag{2.5.1}$$

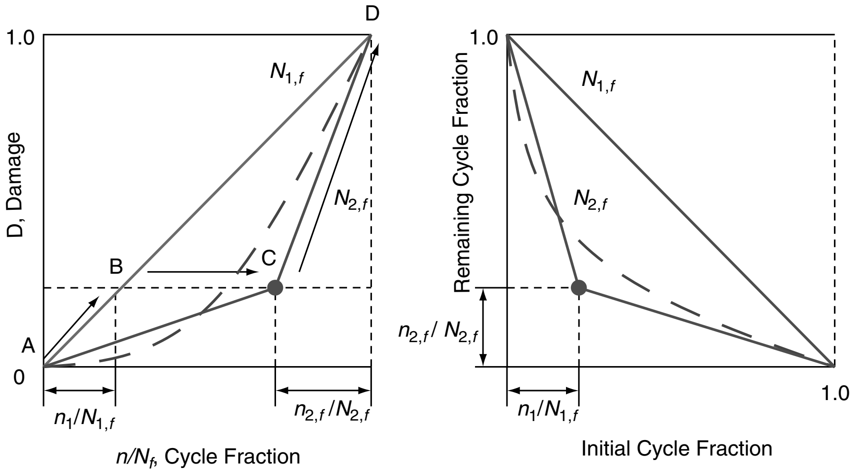

FIGURE 2.10 Double linear damage and damage curve accumulations for two-step loading.

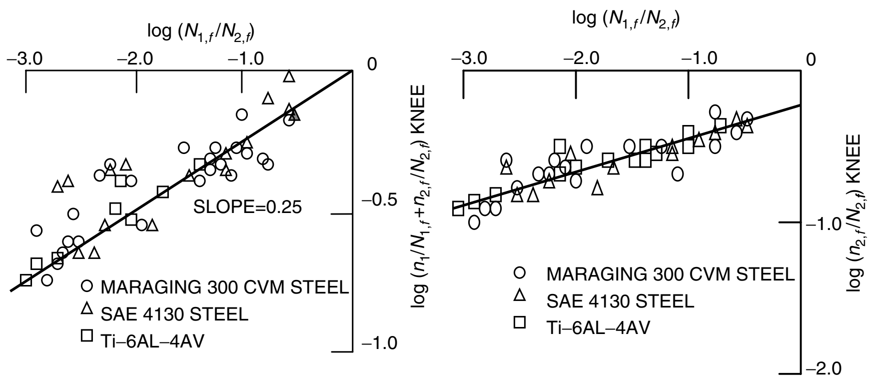

FIGURE 2.11 The knee point coordinates for the double linear damage rule.

$$\left[\frac{n_{2,f}}{N_{2,f}}\right]_{\text{knee}} = 0.65 \times \left(\frac{N_{1,f}}{N_{2,f}}\right)^{0.25} \tag{2.5.2}$$

It is important to note that the knee coordinates are independent of the specific material. Hence, the knee points would be the same for all materials. Their location is dependent only on maximum and minimum lives.

Example 2.3. Block loads are successively applied to a component until failure occurs. Each block has a two-step loading in which 10 cycles of the first loading are applied and followed by 10^3 cycles of the second loading. The first and second loading alone will fail the component to 10^3 cycles and 10^5 cycles, respectively. Determine how many blocks the component can sustain.

Solution. The first step is to construct the double linear damage rule. As shown in the double linear damage accumulation plot (Figure 2.12), a straight line connecting (0,0) with (1,1) is chosen as the reference damage curve for the lower life level ($N_{1,f} = 10^3$ cycles). Along this line, the damage and the coordinate at the breakpoint are defined by Equation 2.5.1, i.e.,

$$D_{\text{knee}} = \left[\frac{n_1}{N_{1,f}}\right]_{\text{knee}} = 0.35 \times \left(\frac{N_{1,f}}{N_{2,f}}\right)^{0.25} = 0.35 \times \left(\frac{10^3}{10^5}\right)^{0.25} = 0.111$$

Therefore, the number of cycles to Phase I damage for the 10^3-cycle-life loading is 111 cycles (i.e., $N_{I1,f} = 0.111 \times 10^3$), and the number of cycles to Phase II damage is 889 cycles (i.e., $N_{II1,f} = 1000 - 111$). The remaining cycle

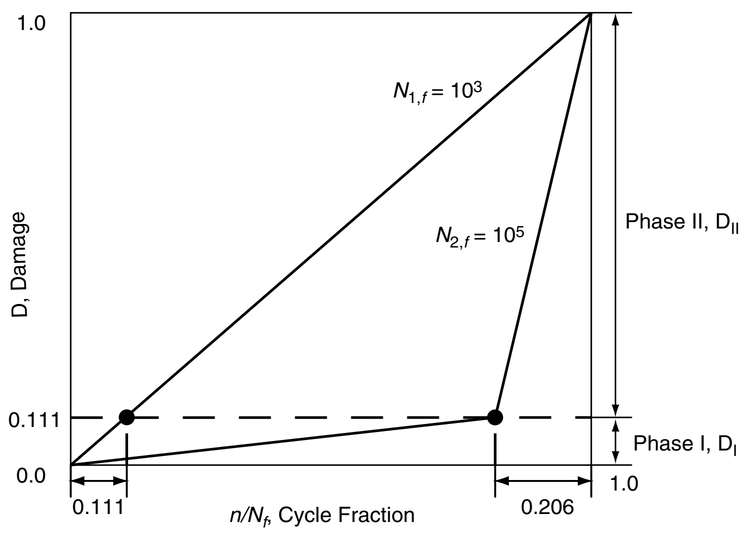

FIGURE 2.12 Double linear damage rule for two-step load levels.

fraction of the break point at the equivalent damage of 0.111 for 10^3-cycle-life loading can be defined by Equation 2.5.2:

$$\left[\frac{n_{2,f}}{N_{2,f}}\right]_{\text{knee}} = 0.65 \times \left(\frac{N_{1,f}}{N_{2,f}}\right)^{0.25} = 0.65 \times \left(\frac{10^3}{10^5}\right)^{0.25} = 0.206$$

The number of cycles to Phase II damage for the 10^5-cycle-life loading is 20,600 cycles (i.e., $N_{II2,f} = 0.206 \times 10^5$), and the number of cycles to Phase I damage is 79,400 cycles (i.e., $N_{I2,f} = 100{,}000 - 20{,}600$).

The next step is to track the damage accumulation as successive loading blocks are applied. To reach Phase I damage the number of cycles for the 10^3-cycle-life loading alone is 111 cycles and for the 10^5-cycle-life loading alone is 79,400 cycles. Therefore, the number of blocks (B_I) to complete Phase I damage can be calculated as follows:

$$B_I \times \left[\frac{10}{111} + \frac{1000}{79400}\right] = 1.0; \quad B_I = 9.7 \text{ blocks}$$

In the similar fashion, the number of blocks (B_{II}) to complete Phase II damage is estimated in the following:

$$B_{II} \times \left[\frac{10}{899} + \frac{1000}{20600}\right] = 1.0; \quad B_{II} = 17.0 \text{ blocks}$$

Thus, based on the double linear damage rule the total blocks to failure ($B_I + B_{II}$) are 26.7 blocks ($= 9.7 + 17.0$). For reference purpose, the Miner linear damage rule would have unconservatively predicted 50 blocks for this sequence of loading.

For block loading involving more than two load levels, the following procedures (Manson and Halford, 1981) for the double linear damage rule were developed. It is conservative to assume that if the individual loading within the block has lives from $N_{\text{low}} = N_{1,f}$ to $N_{\text{high}} = N_{2,f}$, the bilinear damage curve for other loading can be interpolated from the double linear damage rule established by Equations 2.5.1 and 2.5.2. It is stated that the total fatigue life (N_f) can be decomposed into Phase I fatigue life (N_I) and Phase II fatigue life (N_{II}), i.e.,

$$N_f = N_I + N_{II} \tag{2.5.3}$$

One chooses the following form for the relationship between Phase I fatigue life and the total fatigue life:

$$N_I = N_f \exp\left(Z N_f^{\phi}\right) \tag{2.5.4}$$

where Z and ϕ are constants. The two parameters can be determined from the two knee points on the N_I curve for the double linear damage rule. Applying Equations 2.5.1 and 2.5.2 for the knee points leads to the following equations for the number of cycles to Phase I damage for the two loads with the N_{1f} and N_{2f} life levels:

$$N_{I,N_{1,f}} = N_{1,f}\left(\frac{n_1}{N_{1,f}}\right)_{\text{knee}} = 0.35N_{1,f}\left(\frac{N_{1,f}}{N_{2,f}}\right)^{0.25} \tag{2.5.5}$$

$$N_{I,N_{2,f}} = N_{2,f}\left(1-\left(\frac{n_{2,f}}{N_{2,f}}\right)_{\text{knee}}\right) = N_{2,f}\left(1-0.65\left(\frac{N_{1,f}}{N_{2,f}}\right)^{0.25}\right) \tag{2.5.6}$$

Substituting into Equation 2.5.4 allows the solution for Z and ϕ as follows:

$$\phi = \frac{1}{Ln\left(N_{1,f}/N_{2,f}\right)} Ln\left[\frac{Ln(0.35(N_{1,f}/N_{2,f})^{0.25})}{Ln(1-0.65(N_{1,f}/N_{2,f})^{0.25})}\right] \tag{2.5.7}$$

$$Z = \frac{Ln(0.35(N_{1,f}/N_{2,f})^{0.25})}{N_{1,f}^{\phi}} \tag{2.5.8}$$

The equation for N_{II} becomes

$$N_{II} = N_f - N_I = N_f(1-\exp(ZN_f^{\phi})) \tag{2.5.9}$$

Example 2.4. An additional loading with 100 cycles is inserted into the previous two-step loading in Example 2.3. This loading will produce the intermediate fatigue life of 10^4 cycles. Determine how many blocks the component can sustain.

Solution. Because the double linear damage rule based on $N_{\text{low}} = N_{1,f} = 10^3$ cycles and $N_{\text{high}} = N_{2,f} = 10^5$ cycles was constructed in Example 2.3, the interpolation formulas will be used to generate the bilinear damage model for the 10^4-cycle-life loading (as shown in Figure 2.13). From Equations 2.5.7 and 2.5.8, the two parameters Z and ϕ are determined as follows:

$$\phi = \frac{1}{Ln(10^3/10^5)} Ln\left[\frac{Ln\left(0.35\left(10^3/10^5\right)^{0.25}\right)}{Ln\left(1-0.65\left(10^3/10^5\right)^{0.25}\right)}\right] = -0.4909$$

$$Z = \frac{Ln\left(0.35\left(10^3/10^5\right)^{0.25}\right)}{\left(10^3\right)^{-2.077}} = -65.1214$$

Based on Equation 2.5.4, the number of cycles to Phase I damage for the 10^4-cycle-life loading is

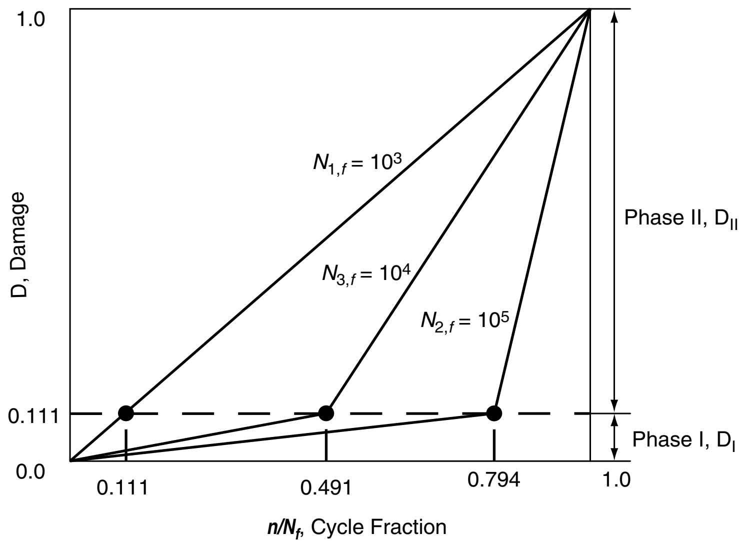

FIGURE 2.13 Double linear damage rule for three-step load levels.

$$N_{I,3,f} = N_f \exp(ZN_f^{\phi}) = 10^4 \exp(-65.1214 \times (10000)^{-0.4904}) = 4910 \text{ cycles}$$

The number of cycles to Phase II damage is then calculated:

$$N_{II,3,f} = N_f - N_I = 10{,}000 - 4909 = 5090 \text{ cycles}$$

Therefore, the number of blocks (B_I) to complete Phase I damage can be calculated as follows:

$$B_I \times \left[\frac{10}{111} + \frac{100}{4910} + \frac{1000}{79400}\right] = 1.0; \quad B_I = 8.1 \text{ blocks}$$

The number of blocks (B_{II}) to complete Phase II damage is estimated in the following:

$$B_{II} \times \left[\frac{10}{899} + \frac{100}{5910} + \frac{1000}{20600}\right] = 1.0; \quad B_{II} = 13.1 \text{ blocks}$$

The total number of blocks to failure is estimated to be 21.2 blocks (= 8.1 + 13.1). The Miner linear damage rule would have unconservatively predicted 33 blocks.

Example 2.5. Repeat Example 2.1 by using the double linear damage rule for fatigue life estimation.

Solution. To construct the double linear damage rule, $N_{low} = N_{1,f} = 10^3$ cycles and $N_{high} = N_{2,f} = 10^6$ cycles were chosen. The bilinear damage models for the 10^4-cycle-life and the 10^5-cycle-life loads are interpolated

TABLE 2.4 Four-Level Step-Stress Fatigue Test Data

S_i (MPa)	n_i (cycles)	$N_{i,f}$ (cycles)	$N_{I,i,f}$ (cycles)	$N_{II,i,f}$ (cycles)	$D_{I,i}$	$D_{II,i}$
±800	10	1000	62	938	0.1613	0.0107
±600	100	10000	3750	6250	0.0267	0.0160
±400	1000	100000	70700	29300	0.0141	0.0341
±200	10000	1000000	883000	117000	0.0113	0.0855
Total					0.2134	0.1463

Note: Blocks to complete Phase I damage = 1/0.2134 = 4.7 blocks; blocks to complete Phase II damage = 1/0.1463 = 6.8 blocks; total blocks to failure = 4.7 + 6.8 = 11.5 blocks.

later with the two calculated parameters ($\phi = -0.4514$ and $Z = -62.7717$). The coordinates of the knee points and the corresponding Phase I and Phase II damage values for four-step step-stress loads are presented in Table 2.4. The blocks required to complete Phase I and Phase II are calculated as 4.7 blocks and 6.8 blocks, respectively, leading to a total of 11.5 blocks to produce failure. When compared with the estimated 11 blocks to failure using the damage curve approach as illustrated in Example 1.1, this shows a very good correlation. The predicted 11.5 blocks to failure by using the double linear damage rule is very close to the prediction (11 blocks) by the damage curve accumulation rule.

2.6 CONCLUSIONS

The double linear damage rule by Manson and Halford is recommended for use in engineering design for durability because of the tedious iteration process by the nonlinear damage theory and the deficiency in linear damage assessment. Note that the knee coordinates in the double linear damage rule are independent of the specific material. Hence, the knee points would be the same for all materials. Their location depends only on the maximum and minimum lives.

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